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Creeping flow around a deforming sphere

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The flow of an incompressible viscous fluid past a deforming sphere is studied for small values of the Reynolds number. The deformation is assumed to be radial but is otherwise quite general. The case of $S = O(1)$, where S is the Strouhal number, is investigated in detail. In particular, the drag is obtained up to $O(R^2 \ln R)$, where R is the Reynolds number.

1. Introduction

Proudman & Pearson (1957) have obtained the steady flow field around a rigid sphere for small Reynolds numbers to the term of $O(R^2 \ln R)$. The analysis has been continued by Chester & Breach (1969) to the term of $O(R^3 \ln R)$. The present work has resulted from the authors' interest in the general problem of unsteady flow around a deforming boundary. Creeping flow around a sphere which deforms in a prescribed manner is considered. The deformation is assumed to be radial but is otherwise quite general. The analysis is carried out to the term of $O(R^2 \ln R)$. Applications of the general results to the cases of a constant rate of deformation and a pulsating sphere are given as examples. Several cases corresponding to Strouhal numbers of different orders of magnitude are discussed.

The flow around a deformable boundary is of considerable theoretical and practical interest. Lighthill (1952), using the Stokes approximation, has investigated the swimming of a tailess object by considering the squirming motion of a deformable sphere in a quiescent fluid. He assumed the deformation to be sufficiently small and gradual that the resulting flow is quasi-steady and that the boundary conditions can be applied at the mean radius of the axisymmetrically pulsating sphere. Another related problem of slow viscous flow around a deforming circular cylinder of infinite length has been solved by Lagerstrom & Cole (1955). The present problem has been solved within the framework of the Stokes approximation by Gautesen & Lin (1971).

2. Basic equations

Consider the flow of a viscous incompressible fluid around a sphere of radius a_1 which deforms radially according to $a_1 = a_0$ for $t_1 \leqslant 0$ and $a_1 = a_1(t_1)$ for $t_1 > 0$, where t_1 denotes time. The governing equations are

$$\frac{\partial \mathbf{q}_1}{\partial t_1} + (\mathbf{q}_1 . \nabla_1)\mathbf{q}_1 = \nu \nabla_1^2 \mathbf{q}_1 - \frac{1}{\rho_0}\nabla_1 p_1,$$
$$\nabla_1 . \mathbf{q}_1 = 0,$$

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where ρ_0 is the density, ν the kinematic viscosity and $\mathbf{q}_1$ the velocity. Let the characteristic velocity U be the free-stream velocity and let the maximum radial velocity W of the boundary be of $O(U)$. If the change in velocity of order W takes place over a distance of order a_0 and over a time period of order a_0/W, then a_0 and a_0/W are the characteristic length and time respectively. If the pressure variation in the vicinity of the sphere is of the same order of magnitude as the viscous stresses, then p_1 is characterized by $\rho_0 \nu U/a$. Non-dimensionalization of the above equations by these characteristic quantities gives

$$\left.\begin{aligned} SR\,\partial\mathbf{q}/\partial t + R(\mathbf{q}\,.\nabla)\mathbf{q} &= \nabla^2\mathbf{q} - \nabla p,\\ \nabla.\mathbf{q} &= 0,\end{aligned}\right\} \tag{2.1}$$

where the unsubscripted letters stand for the same physical quantities as their subscripted counterparts defined above and $R = Ua_0/\nu$ and $S = W/U$ are the Reynolds and the Strouhal numbers respectively.

The stream function ψ satisfies

$$q_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad q_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

in spherical co-ordinates (r, θ, ϕ). In terms of ψ, equation (2.1) becomes

$$SR\frac{\partial}{\partial t}D^2\psi - D^4\psi = -R\left[\frac{1}{r^2}\frac{\partial(\psi, D^2\psi)}{\partial(r,\mu)} + \frac{2}{r^2}(D^2\psi)(L\psi)\right], \tag{2.2}$$

where $\mu = \cos\theta$,

$$\left.\begin{aligned} D^2 &= \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2}\\ \text{and}\quad L &= \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\mu},\end{aligned}\right\} \tag{2.3}$$

In the region where $r = O(R^{-1})$ the velocity and the pressure vary respectively by $O(U)$ and $O(\rho_0 U^2)$ over a large distance of $O(a_0/R)$. Thus the appropriate normalization factors for length, velocity and pressure are respectively a_0/R, U and $\rho_0 U^2$. The time and the distance are stretched by the same factor R in this region, i.e. $\tau = Rt$ and $\rho = Rr$. Thus in terms of $\Psi = R^2\psi$, equation (2.1) becomes

$$\frac{\partial}{\partial\tau_1}D_\rho^2\Psi - D_\rho^4\Psi = -\frac{1}{\rho^2}\frac{\partial(\Psi, D_\rho^2\Psi)}{\partial(\rho,\mu)} - \frac{2}{\rho^2}(D_\rho^2\Psi)(L_\rho\Psi), \tag{2.4}$$

where $\tau_1 = \tau/S$ and D_ρ^2 and L_ρ are the same operators as those given in (2.2), but with r replaced by ρ.

We apply the method of matched asymptotic expansions, which is described in the next section, to obtain a solution to the problem under consideration. The inner solution must satisfy (2.2),

$$q_r(a,\theta) = S\,da/dt, \quad q_\theta(a,\theta) = 0$$

and agree with the known steady flow for $t \leqslant 0$ (before the sphere deforms). The outer solution must satisfy (2.4), approach uniform flow at infinity and agree with the known steady solution in the outer region for $t \leqslant 0$. The inner and outer solutions are then matched at small ρ.

3. Matched asymptotic solutions

3.1. *The leading terms of the expansions*

In this and the following two sections only the case $S = O(1)$ is considered. Other cases are discussed in the last section. Expansion of the solution to (2.4) for small R and fixed ρ gives the outer (Oseen) expansion

$$\Psi = \Psi_0(\tau_1, \rho, \mu) + F_1(R)\,\Psi_1(\tau_1, \rho, \mu) + F_2(R)\,\Psi_2(\tau_1, \rho, \mu) + \ldots,$$

where $F_{n+1}(R)/F_n(R) = o(R)$.

The first term of the above expansion is the known uniform stream

$$\Psi_0 = \tfrac{1}{2}\rho^2(1-\mu^2).$$

Since the Reynolds number is defined using a_0, the expansion for small R is valid only if the radius of the sphere remains of $O(a_0)$ during the deformation. We assume that, whenever a large acceleration takes place, it does so over a time period small enough that S remains of $O(1)$ during the deformation. Further normalization of time as $\sigma = t/SR = t_1(a_0^2/\nu)$ and expansion of the solution to (2.2) for small R and fixed r gives the inner (Stokes) expansion

$$\psi = \psi_0(\sigma, r, \mu) + f_1(R)\,\psi_1(\sigma, r, \mu) + f_2(R)\,\psi_2(\sigma, r, \mu) + \ldots,$$

where $f_{n+1}(R)/f_n(R) = o(R)$. We emphasize that the time normalization

$$\sigma = t_1/(a_0^2/\nu)$$

does not imply that our interest is confined to $\sigma = O(1)$, i.e. $t = O(R)$.

Substitution of the inner expansion into (2.2) gives to first order

$$D^2\left(\frac{\partial}{\partial\sigma} - D^2\right)\psi_0 = 0. \tag{3.1}$$

The solution is

$$\psi_0 = Q_1(\mu)\,[E(\sigma)r^2 + Gr + H(\sigma)r^{-1} + Fr^4] + \frac{Q_1(\mu)}{r}\int_{a(\sigma)}^{r} sX_0(s,\sigma)\,ds + \mu g(\sigma), \tag{3.2}$$

where the last term, which is a point source, arises from the radial deformation of the sphere, the term in brackets corresponds to the steady Stokes solution and the integral term is the rotational part of the unsteady (non-quasi-steady) solution. The variable X_0 in (3.2) satisfies the heat equation

$$\partial^2 X_0/\partial r^2 = \partial X_0/\partial\sigma, \tag{3.3}$$

and $Q_1(\mu) = \frac{1}{2}(\mu^2 - 1)$ is the first of a sequence of polynomials defined as integrals of Legendre polynomials:

$$Q_n(\mu) = \int_{-1}^{\mu} P_n(\mu)\,d\mu, \quad (n \geqslant 1).$$

Application of the boundary and initial conditions and the condition that $R^2\psi_0$ matches Ψ_0 yields

$$\left.\begin{aligned} &E(\sigma) = -1, \quad G = \tfrac{3}{2}, \quad F = 0, \quad H(\sigma) = a^3 - \tfrac{3}{2}a^2, \quad g(\sigma) = -a^2\dot{a}/R, \\ &X_0(r,0) = 0, \quad X_0(a,\sigma) = 3(a-1), \quad X_0(\infty,\sigma) = O(1), \end{aligned}\right\} \tag{3.4}$$

where $\dot{a} = da/d\sigma$. The condition on g in (3.4) reflects the restriction that $S = O(1)$. For this case $da/dt = O(1)$ or $da/d\sigma = SR(da/dt) = O(R)$. Therefore $g(\sigma) = O(1)$ and ψ_0 remains of $O(1)$. We now show that $X_0 = 3(a-1)+O(R)$. We write

$$X_0 = X_{00} + RX_{01} + \dots,$$

where X_{00} is the solution for $R = 0$ (i.e. quasi-steady solution). Thus

$$\partial^2 X_{00}/\partial^2 r = 0, \text{ with } X_{00}(a,\sigma) = 3(a-1) \text{ and } X_{00}(\infty,\sigma) = O(1),$$

which has the unique solution $X_{00}(r,\sigma) = 3(a-1)$. Now X_{01} satisfies

$$\frac{\partial^2 X_{01}}{\partial r^2} - \frac{\partial X_{01}}{\partial \sigma} = \frac{1}{R}\frac{\partial X_{00}}{\partial \sigma} = \frac{3\dot{a}}{R} = O(1).$$

Thus $X_{01} = O(1)$ and we achieve $X_0 = 3(a-1)+O(R)$. It then follows that $\partial X_0/\partial\sigma = 3\dot{a}+O(R) = O(R)$. With this result we deduce from (3.1) and (3.3) that the transient viscous diffusion of vorticity associated with radial deformation is of order R when the sphere translates at a speed such that $S = O(1)$ and $R \ll 1$. On the other hand, the same radial deformation produces larger inertial effects given by the last term of (3.2), if $\ddot{a}(\sigma) > O(R)$. In this first-order solution as well as in solutions to any other order the boundary conditions are satisfied exactly rather than to the appropriate order. This is necessary for obtaining a uniformly valid transient solution for each order. The order of magnitude of the transient solution may change with time according to the changes in $da/d\sigma$, although at no time should it exceed order one.

Since $X_0 = 3(a-1)+O(R)$, to order R we may take $X_0 = 3(a-1)$. This leads to

$$\psi_0 = Q_1(\mu)[-r^2 + \tfrac{3}{2}ar - a^3/2r] - \mu a^2\dot{a}/R, \tag{3.5}$$

which is just the quasi-steady Stokes solution. We remark that the same result could be obtained directly by allowing G to depend on σ with the condition $\dot{G}(\sigma) = O(R)$. This approach is taken in the higher order solutions.

In the outer region

$$R^2\psi_0 \sim -\tfrac{1}{2}Q_1[2\rho^2 - 3Ra\rho + R^3a^3/\rho] - Ra^2\dot{a}\mu.$$

Therefore, the highest order unsteady term of the Oseen solution in the Stokes region is of $O(R^2)$. The second term of the Oseen expansion with $F_1(R) = R$ is expressed as $\Psi_1 = \Psi_{1s} + \Psi_{1u}$, where Ψ_{1s} is the known steady solution,

$$\Psi_{1s} = -\tfrac{3}{2}(1+\mu)\,[1-\exp\{-\tfrac{1}{2}\rho(1-\mu)\}],$$

and Ψ_{1u} is the unsteady part of the solution, which satisfies

$$\left(\frac{\partial}{\partial\tau_1} - D_\rho^2\right) D_\rho^2 \Psi_{1u} = -\left(\frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu} + \mu\frac{\partial}{\partial\rho}\right) D_\rho^2 \Psi_{1u}.$$

The transformation $D_\rho^2\Psi_{1u} = \exp[\tfrac{1}{2}\rho\mu]\,\Phi$ reduces the above equation to

$$(\partial/\partial\tau_1 + \tfrac{1}{4} - D_\rho^2)\,\Phi = 0.$$

It follows from the above equation, from the condition of vanishing vorticity at infinity

$$D_\rho^2\Psi_{1u} = 0 \quad \text{as} \quad \rho \to \infty,$$

from the matching condition on the vorticity

$$D_\rho^2 \Psi_{1u} = D^2 \psi_{0u} = 0 \quad \text{as} \quad \rho \to 0,$$

and from the initial condition

$$D_\rho^2 \Psi_{1u} = 0 \quad \text{at} \quad \sigma = 0$$

that $D_\rho^2 \Psi_{1u} = 0$. Thus in the second term of the Oseen expansion, time only appears as a parameter. Hence,

$$\Psi_1 = -\tfrac{3}{2}a(1+\mu)\left[1-\exp\left(-\tfrac{1}{2}\rho(1-\mu)\right)\right] + \left[A(\tau_1)\rho^2 + B(\tau_1)\rho^{-1}\right]Q_1(\mu).$$

Here, B is zero; otherwise it leads to an unbounded term of $O(1/R^2)$ in the inner region. A will be shown to vanish when $R\Psi_1$ is matched with $R^2(R\psi_1)$.

3.2. *The second term of the inner expansion*

We now obtain ψ_1. Substitution of the inner expansion into (2.2) shows that $f_1(R) = R$ and

$$D^2\left(\frac{\partial}{\partial\sigma} - D^2\right)\psi_1 = -\frac{9}{2}\left(\frac{2a}{r^2} - \frac{3a^2}{r^3} + \frac{a^4}{r^5}\right)Q_2 - \frac{3\dot{a}}{R}\left(\frac{3a^3}{r^4} - \frac{1}{r}\right)Q_1, \tag{3.6}$$

where $Q_2(\mu) = -\tfrac{1}{2}\mu(1-\mu^2)$. The solution to (3.6) is sought in the form

$$\psi_1 = Q_1\psi_{11} + Q_2\psi_{12},$$

which upon substitution into (3.6) yields

$$\left(\frac{\partial}{\partial\sigma} - \frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\right)\psi_{11} = \dot{K}(\sigma)r^2 + \dot{L}(\sigma)r^{-1} - \frac{3}{2}\frac{\dot{a}}{R}\left(r + \frac{3}{2}\frac{a^3}{r^2}\right), \tag{3.7}$$

$$\left(\frac{\partial}{\partial\sigma} - \frac{\partial^2}{\partial r^2} + \frac{6}{r^2}\right)\psi_{12} = \dot{N}(\sigma)r^{-2} - \tfrac{3}{8}a\left(-4 + 9\frac{a}{r} + 2\left(\frac{a}{r}\right)^3\right). \tag{3.8}$$

In arriving at (3.8) the term containing r^3 was omitted since it leads to an unbounded term in the Oseen region. The solutions to (3.7) and (3.8) are required to satisfy separately the conditions that the velocity perturbation vanishes on the deforming boundary and that the known steady solution is recovered for $t \leqslant 0$, i.e.

$$\psi_{1i}(a,\sigma) = 0, \quad \partial\psi_{1i}/\partial r(a,\sigma) = 0 \quad \text{for} \quad (i = 1, 2),$$
$$\psi_{11}(r,0) = -\tfrac{3}{16}(2r^2 - 3r + r^{-1}),$$
$$\psi_{12}(r,0) = \tfrac{3}{16}(2r^2 - 3r + 1 - r^{-1} + r^{-2}),$$

where the initial values for ψ_{11} and ψ_{12} have been obtained from Proudman & Pearson (1957). The solutions to (3.7) and (3.8) which lead to bounded velocity fields are

$$\left.\begin{aligned} \psi_{11} &= K(\sigma)r^2 + C(\sigma)r^{-1} + Pr + r^{-1}\int_{a(\sigma)}^{r} sX(s,\sigma)\,ds + h(\sigma), \\ \psi_{12} &= \tfrac{3}{8}ar^2 - \tfrac{9}{16}a^2 r + M(\sigma) - \tfrac{3}{16}a^4 r^{-1} + D(\sigma)r^{-2}, \end{aligned}\right\} \tag{3.9}$$

where X satisfies

$$\frac{\partial^2 X}{\partial r^2} - \frac{\partial X}{\partial\sigma} = \frac{3\dot{a}}{R}\left(1 - \frac{3}{4}\frac{a^3}{r^3}\right) = G_0(r,\sigma). \tag{3.10}$$

We remark that ψ_{11} satisfies (3.7) exactly, while ψ_{12} satisfies (3.8) to order R under assumption that $\dot{M}(\sigma) = O(R)$. In addition, L and N were renamed in arriving at (3.9) to include some additional functions of σ. Matching gives $K(\sigma) = -\frac{3}{8}a$ and $A(\tau_1) = 0$. Application of the boundary and initial conditions to ψ_{11} and ψ_{12} gives

$$C(\sigma) = -\tfrac{3}{16}a^4, \quad P = \tfrac{9}{16}a^2,$$
$$X(r,0) = 0, \quad X(a,\sigma) = 0, \quad X(\infty,\sigma) = O(1),$$
$$D(\sigma) = \tfrac{3}{16}a^5, \quad M(\sigma) = \tfrac{3}{16}a^3.$$

Note that $\dot{M}(\sigma) = O(R)$ and that we have chosen $X(a,\sigma) = 0$, which makes X unique (the boundary conditions then determine C and P). We take P to be a function of time so that the boundary conditions are satisfied exactly. This choice of P ensures that (3.7) is satisfied to the appropriate order. A similar situation was encountered in the determination of G in ψ_0. The solution of (3.10) can be obtained by use of a Green's function. Let

$$x = r - a(\sigma) \quad \text{and} \quad X(r,\sigma) = Y(x,\sigma).$$

Then (3.10) can be rewritten as

$$Y_{xx} = -Y_\sigma = G_0(x+a,\sigma) - \dot{a}(\sigma)Y_x \quad (0 \leqslant \sigma,\ 0 \leqslant x).$$

The corresponding boundary and initial conditions are

$$Y(0,\sigma) = 0 \quad \text{and} \quad Y(x,0) = 0.$$

The solution of the above equation with the $O(R)$ term $\dot{a}(\sigma)Y_x$ neglected leads to

$$X(r,\sigma) = -\frac{1}{2}\int_0^\sigma \int_{a(\tau)}^\infty \frac{G_0(r_0,\tau)}{[\pi(\sigma-\tau)]^{\frac{1}{2}}}\left\{\exp\left[-\frac{(r-r_0+a(\tau)-a(\sigma))^2}{4(\sigma-\tau)}\right] - \exp\left[-\frac{(r+r_0-a(\tau)-a(\sigma))^2}{4(\sigma-\tau)}\right]\right\} dr_0\, d\tau.$$

It follows from the above and the definition of G_0 that $X_\sigma(r,\sigma) = O(1)$ as was assumed. We remark that the transient part of the solution is decoupled from the quasi-steady part. While the former remains of the same order for all time the latter changes its order of magnitude with time but never exceeds $O(1)$. Each term ψ_i in our asymptotic expansion can be considered to consist of a transient term ψ_{i1} and a quasi-steady term ψ_{i2}. For our expansion to be valid each term ψ_i should be of $O(1)$. Thus ψ_{i1} and ψ_{i2} need not each be of $O(1)$; only their sum need be. The fact that the transient term changes its order of magnitude then does not invalidate our asymptotic expansion. As long as X_σ is of $O(1)$ for some time, X must be retained in the analysis.

3.3. *The third term in the inner expansion*

We now obtain the third term in the inner expansion. It is known that the third term in the steady solution is of $O(R^2 \ln R)$ (see Proudman & Pearson 1957). Since the unsteady term is at most of the same order of magnitude as the steady term in the inner region we have

$$f_2(R) = R^2 \ln R$$

and

$$D^2(\partial/\partial\sigma - D^2)\psi_2 = 0.$$

The above equation is identical to that which governs ψ_0 and thus ψ_2 is of the same form as ψ_0, i.e.

$$\psi_2 = Q_1(\mu)\,[Y(\sigma)r^2 + Tr + U(\sigma)r^{-1}].$$

Now the term $Y(\sigma)r^2$ leads to a term $Y(\sigma)(R^2\ln R)\rho^2$ in the Oseen region. However, there is no such term in that region, since such an Oseen term requires in the Stokes region a $\ln R$ term which is unbounded as $R\to 0$. Therefore Y must be so chosen that this term is cancelled by a similar term in $R^2\psi_3$. Substitution of the inner expansion into (2.2) and solution of the resulting equation to $O(R^2)$ shows that the only term in $R^2\psi_3$ which can cancel $Y(R^2\ln R)\rho^2$ is $-\frac{9}{40}a^2R^2r^2\ln(r/a)$. This term when expressed in the Oseen variables becomes

$$\tfrac{9}{40}a^2(R^2\ln R)\rho^2 - \tfrac{9}{40}a^2R^2\rho^2\ln\rho + \tfrac{9}{40}a^2(R^2\ln a)\rho^2,$$

from which its follows that $Y(\sigma) = -\frac{9}{40}a^2$. The boundary condition that the velocity perturbation is zero on the moving boundary and the initial condition

$$\psi_2(r,\mu,0) = -\tfrac{9}{80}(2r^2 - 3r + r^{-1})\,Q_1(\mu)$$

then give
$$T = \tfrac{27}{80}a^3, \quad U(\sigma) = -\tfrac{9}{80}a^5.$$

The argument which was used in the determination of P and G applies equally to T.

4. Drag force

The following drag computation follows closely the line of Chester & Breach (1969). In this section the pressure and the derivatives of all quantities appearing are understood to be evaluated at the surface of the sphere ($r = a$). Integration of the stresses over the entire deforming surface gives

$$\begin{aligned} D &= 2\pi\rho\nu a_0 U\int_0^\pi \left[\left(-p + 2\frac{\partial q_r}{\partial r}\right)\cos\theta - \left(\frac{\partial p_\theta}{\partial r} - \frac{q_\theta}{r} + \frac{1}{r}\frac{\partial q_r}{\partial\theta}\right)\sin\theta\right] a^2\sin\theta\, d\theta \\ &= \frac{D_s}{3}\int_0^\pi \left[-p\cos\theta + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\right] a^2\sin\theta\, d\theta, \end{aligned} \qquad (4.1)$$

in which D_s is the Stokes drag on a sphere of radius a_0 and p is given by the θ component of (2.1), i.e.

$$a\frac{\partial q_\theta}{\partial\sigma} + R^{-1}a\dot{a}\frac{\partial q_\theta}{\partial r} = -\frac{\partial p}{\partial\theta} + \frac{\partial^2(rq_\theta)}{\partial r^2}.$$

Integration of the above equation with respect to θ when q_θ is expressed in terms of ψ yields

$$p - \bar{p} = \int^\theta \left[\dot{a}\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^3\psi}{\partial r^3} + \frac{\partial^2\psi}{\partial\sigma\,\partial r}\right]\frac{d\theta}{\sin\theta}. \qquad (4.2)$$

Now, the inner solution can be written as

$$\psi = \sum_{n=1}^{\infty} Q_n(\mu)\,\Phi_n(r) - a^2\dot{a}\mu/R,$$

which, together with (4.1) and (4.2), gives

$$p-\bar{p} = \sum_{n=1}^{\infty}\left[\dot{a}\left(\frac{\partial^2\Phi_n}{\partial r^2}-\frac{1}{a}\frac{\partial\Phi_n}{\partial r}\right)-\frac{\partial^3\Phi_n}{\partial r^3}+\frac{\partial^2\Phi_n}{\partial\sigma\,\partial r}\right]\frac{P_n(\mu)}{n(n+1)}$$

and
$$D = \frac{a^2D_s}{3}\left[\sum_{n=1}^{\infty}\left\{\dot{a}\left(\frac{\partial^2\Phi_n}{\partial r^2}-\frac{1}{a}\frac{\partial\Phi_n}{\partial r}\right)-\frac{\partial^3\Phi_n}{\partial r^3}+\frac{\partial^2\Phi_n}{\partial\sigma\,\partial r}\right\}\int_1^{-1}\frac{P_n(\mu)}{n(n+1)}\mu\,d\mu - \sum_{n=1}^{\infty}\frac{\partial^2\Phi_n}{\partial r^2}\frac{1}{a}\int_1^{-1}Q_n(\mu)\,d\mu\right]$$
$$= \frac{a^2}{9}D_s\left[-\left(\dot{a}+\frac{2}{a}\right)\frac{\partial^2\Phi_1}{\partial r^2}+\frac{\partial^3\Phi_1}{\partial r^3}-\frac{\partial^2\Phi_1}{\partial\sigma\,\partial r}\right].$$

The inner expansion and the above equation yield

$$D = D_s[a+a^2R\{\tfrac{3}{8}+\tfrac{1}{9}G_0(a,\sigma)-(3a)^{-1}X_r(a,\sigma)\}+R^2\ln R(\tfrac{9}{40}a^3)]. \tag{4.3}$$

For the case of a rigid sphere, $a = 1, \dot{a} = 0, 1, X = 0$ and $G(a,\sigma) = 0$ and the above drag reduces to the known result obtained by Proudman & Pearson.

5. Examples

5.1. *A pulsating sphere*

Consider a sphere pulsating according to $a(\sigma) = 1+\alpha^2 f(\sigma/\alpha)$, where $\alpha = O(R)$. Then $G_0(r,\sigma) = 3\alpha R^{-1}f'(\sigma/\alpha)[1-3a^3/(4r^3)]$ and

$$X_r(a,\sigma) = -\frac{3\alpha}{4\pi^{\frac{1}{2}}R}\left[\int_0^{\sigma}\frac{f'(\tau/\alpha)}{(\sigma-\tau)^{\frac{1}{2}}}d\tau+9\int_0^{\sigma}\int_1^{\infty}\frac{f'(\tau/\alpha)}{(\sigma-\tau)^{\frac{1}{2}}}\exp\left[-\frac{(r_0-1)^2}{4(\sigma-\tau)}\right]\frac{dr_0}{r_0^4}d\tau\right],$$

where primes denote differentiation. It can be shown that the second integral is of $O(\alpha)$. Note that $da(\sigma)/d\sigma = O(R)$, i.e. $da(t)/dt = O(1)$. Hence the present theory for $S = O(1)$ applies. As a specific example, let $f(\tau/\alpha) = 1-\cos(\sigma/\alpha)$. Then the first integral can be reduced to tabulated functions. Since

$$\int_0^{\sigma}\frac{f'(\tau/\alpha)}{(\sigma-\tau)^{\frac{1}{2}}}d\tau = 2\alpha^{\frac{1}{2}}\int_0^{(\sigma-\alpha)^{\frac{1}{2}}}f'\left(\frac{\sigma}{\alpha}-\tau^2\right)d\tau,$$

then

$$X_r(a,\sigma) = -(3/R)(\tfrac{1}{2}\alpha)^{\frac{3}{2}}[\sin(\sigma/\alpha)\,C_1((\sigma/\alpha)^{\frac{1}{2}})-\cos(\sigma/\alpha)\,S_1((\sigma/\alpha)^{\frac{1}{2}})]+O(\alpha),$$

where C_1 and S_1 are the tabulated cosine and sine integrals (Abromowitz & Stegun 1964, pp. 300, 322). The numerical results for the case of $\alpha = 0{\cdot}2$ and $R = 0{\cdot}2$ are given in figure 1. The drag ratio D/D_s is plotted against the phase angle of the pulsation $\theta = \sigma/\alpha$. The results for the corresponding quasi-steady case are also plotted for reference. It is interesting to note that at $\theta = \frac{1}{2}\pi$ and $\theta = \frac{3}{2}\pi$, which correspond to the same radius, the drag ratios are 1·199 and 1·068 respectively. It is no surprise that the drag force on an expanding sphere is 13 per cent larger than that on a shrinking sphere of the same radius. We also note that the drag ratio decreases from 1·061 to 1·030 in one cycle. This drag reduction during the first cycle is due to the fact that the fluid is viscous and cannot respond instantly to the deformation of the sphere. When the sphere is ready to repeat the same deformation at the end of the first cycle, the flow has not

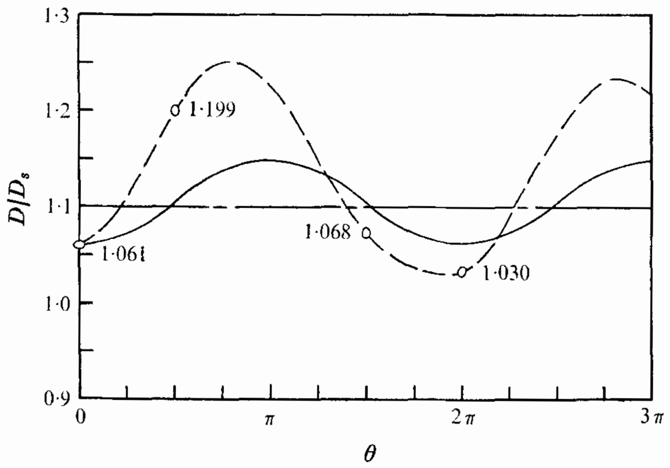

FIGURE 1. Drag force on a pulsating sphere; ——, quasi-steady drag; – – –, total drag.

yet returned to the original uniform stream. However, as time proceeds, the fluid will eventually catch up with the deformation and pulsate relative to the uniform stream with the same frequency. This can be seen from the expressions for $G_0(a,\sigma)$ and $X_r(a,\sigma)$. As $\sigma \to \infty$,

$$D \to D_s[1 + \tfrac{3}{8}R + \tfrac{1}{12}\alpha \sin(\sigma/\alpha) - \tfrac{9}{4}\alpha^{\frac{3}{2}} \sin(\sigma/\alpha - \tfrac{1}{4}\pi) + \tfrac{9}{40}R^2 \ln R + O(R^2)].$$

Hence the drag becomes periodic with frequency $1/\alpha$ for large σ.

5.2. *A constant expansion*

Consider the case $a(t) = 1 + \beta t$, where $\beta = O(1)$, so that $S = O(1)$. In the stretched co-ordinates $a(\sigma) = 1 + \alpha\sigma$, where $\alpha = SR = O(R)$. It follows from the results of §4 that

$$G_0(r,\sigma) = \frac{3\alpha}{R}\left(1 - \frac{3}{4}\frac{a^3}{r^3}\right)$$

and
$$X_r(a,\sigma) = \frac{3\alpha}{2R\pi^{\frac{1}{2}}}\int_0^\sigma \int_{a(\tau)}^\infty \frac{r_0 - a(\tau)}{(\sigma-\tau)^{\frac{3}{2}}} \exp\left[-\frac{(r_0-a)^2}{4(\sigma-\tau)}\right]\left(1 - \frac{3a^3}{4r_0^3}\right) dr_0\, d\tau$$

$$= -\frac{3\alpha}{2R\pi^{\frac{1}{2}}}(4\sigma^{\frac{1}{2}} - \tfrac{3}{4}a^3\bar{I}),$$

where
$$\bar{I} = \int_0^\infty \frac{2\pi^{\frac{1}{2}}}{(r_0+a)^3} \operatorname{erfc}\left(\frac{r_0}{2\sigma^{\frac{1}{2}}}\right) dr_0 + O(\alpha)$$

and
$$\operatorname{erfc}(x) = \frac{2}{\pi^{\frac{1}{2}}}\int_x^\infty e^{-s^2}\, ds.$$

We can simplify $\bar{I}$ to

$$\bar{I} = \frac{\pi^{\frac{1}{2}}}{a^2} - \frac{1}{a\sigma^{\frac{1}{2}}} + \frac{\pi^{\frac{1}{2}}}{2\sigma} - \frac{a}{2}\sigma^{-\frac{3}{2}}\left[\pi^{\frac{1}{2}}F\left(\frac{a}{2\sigma^{\frac{1}{2}}}\right) - \frac{E_i}{2}\left(\frac{a^2}{4\sigma}\right)e^{-a^2/4\sigma}\right],$$

where $E_i(x)$ and $F(x)$ are the exponential integral and the Dawson integral respectively. These integrals are defined and tabulated in Abramowitz & Stegun

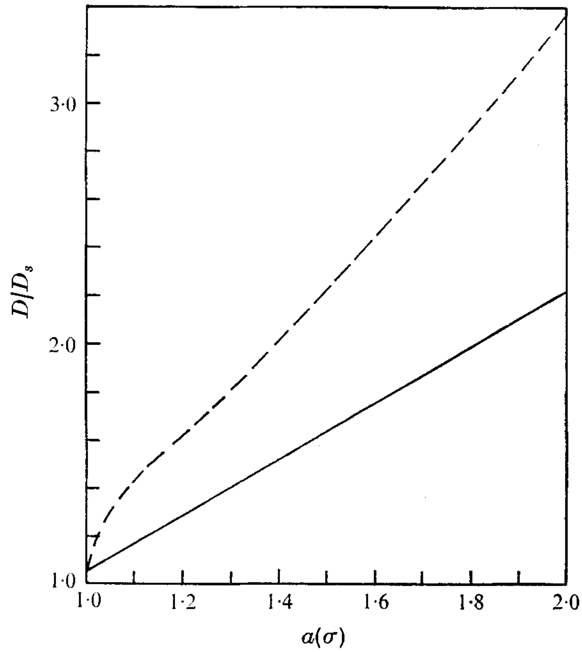

FIGURE 2. Drag force on an expanding sphere; ——, quasi-steady drag; – – –, total drag.

(1964, pp. 228, 298). The numerical results for the case of $\alpha = 0{\cdot}2$ and $R = 0{\cdot}2$ are plotted in figure 2. The results for the corresponding quasi-steady case are also plotted in the same figure for comparison. The significant increase in the drag force due to the radial expansion is obvious. The indicated increase in drag is entirely due to the transient momentum transfer through viscosity.

6. Discussion

The results obtained above are valid as long as the radius of the sphere remains of the same order of magnitude during the deformation. The time rate of deformation has been assumed to be of the same order of magnitude as the ambient uniform stream velocity, i.e. $S = O(1)$. The acceleration $\ddot{a}(t_1)$ is allowed to be larger than $O(\nu U/a_0^2)$, whenever its duration is $t_1 < O(a_0^2/\nu)$, in order that S remains $O(1)$. For the case of $S \gg 1$, the convective acceleration term is no longer negligible and the full Navier–Stokes equation must be solved. On the other hand, if the deformation takes place so gradually that $S \ll 1$, i.e. $\dot{a}(\sigma) < O(R)$ or $\dot{a}(t_1) < O(U)$, then the flow around the sphere is quasi-steady and time enters into the solution only as a parameter. The drag on a deforming sphere in such a quasi-steady flow is given by the results for the case of rigid sphere but with the Reynolds number based on the instantaneous radius. Our procedure can be continued to obtain higher order terms, but the solution to the unsteady Oseen equation with a source term must be obtained. As far as the authors are aware, no precise measurements of the drag force on a deforming sphere are yet available.

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